

CLOSABILITY PROPERTY OF OPERATOR ALGEBRAS GENERATED BY NORMAL OPERATORS AND OPERATORS OF CLASS C_0

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ABSTRACT. An operator algebra \mathcal{A} acting on a Hilbert space is said to have the closability property if every densely defined linear transformation commuting with \mathcal{A} is closable. In this paper we study the closability property of the von Neumann algebra consisting of the multiplication operators on $L^2(\mu)$, and give necessary and sufficient conditions for a normal operator N such that the von Neumann algebra generated by N has the closability property. We also give necessary and sufficient conditions for an operator T of class C_0 such that the algebra generated by T in the weak operator topology and the algebra $H^\infty(T) = \{u(T) : u \in H^\infty\}$ have the closability property.

1. INTRODUCTION

An operator algebra \mathcal{A} with the property that every densely defined linear transformation in the commutant of \mathcal{A} is closable is said to have the closability property. The closability property problem which has close connection with transitive algebra problem was first studied by W. Arveson. In [4], Arveson showed that the algebra L^∞ acting on the Hilbert space L^2 and the algebra H^∞ acting on the Hardy space H^2 have the closability property. In [2], H. Bercovici, R.G. Douglas, C. Foias, and C. Pearcy showed that algebras \mathcal{W}_S and $\mathcal{W}_{S(\theta)}$ generated by the unilateral shift S and the Jordan block $S(\theta)$ (see Section 2 for precise definitions) in the weak operator topology, respectively, and any maximal abelian selfadjoint subalgebra with a cyclic vector have the closability property. They introduced some general viewpoints, such as rationally strictly cyclic vector and confluence for an algebra \mathcal{A} , to determine whether \mathcal{A} has the closability property. They also showed that if an algebra \mathcal{A}_1 is a quasiasaffine transform of an algebra \mathcal{A}_2 which has the closability property then \mathcal{A}_1 has the closability property as well. As a consequence, every unital commutative algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ with a rationally strictly cyclic vector and the commutant of any operator of class C_0 have the closability property. In particular, the algebra $H^\infty(S(\theta)^*) = \{u(S(\theta)^*) : u \in H^\infty\}$ has the closability property for any nonconstant inner function θ which was proved independently by D. Sarason (see [7]). We refer the reader to [2] for further details about confluent operator algebras and the effect of quasimilarity on the closability property.

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In [2, Proposition 3.5], some examples of operator algebras without the closability property were given which point out that an algebra with the closability property must be sufficiently large and should not have uniform infinite multiplicity. This observation motivates us investigate the relation between the closability property and uniform finite multiplicity in this paper.

This paper is organized as follows. In Section 2 we give the preliminaries and terminology of multiplication operators and operators of class C_0 . In Section 3 we study the closability property of the von Neumann algebras \mathcal{A}_μ consisting of the multiplication operators on $L^2(\mu)$ and \mathcal{W}_N^* generated by the normal operator N . We show that $\mathcal{A}_\mu^{(n)}$ has the closability property if and only n is finite while \mathcal{W}_N^* has the closability property if and only if N has uniform finite multiplicity. In the study of the closability property, it is essential to examine closed unbounded linear transformations in the commutant of a bounded operator. In Section 4 we characterize the closed, densely defined linear transformations intertwining two operators of class C_0 . In Section 5 we deal with the closability property of unital algebras $H^\infty(T) = \{u(T) : u \in H^\infty\}$ and \mathcal{W}_T which is generated by T in the weak operator topology where T is an operator of class C_0 , and show that $H^\infty(T)$ has the closability property if T has finite multiplicity. We also provide necessary and sufficient conditions for T with infinity multiplicity such that the algebra $H^\infty(T)$ has the closability property. Moreover, we show that the algebra \mathcal{W}_T has the closability property if and only if the algebra $H^\infty(T)$ does.

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2. PRELIMINARY

Throughout this paper, the Hilbert space is over the complex number \mathbb{C} . The space of bounded linear operators $T : \mathcal{H} \rightarrow \mathcal{K}$, where \mathcal{H} and \mathcal{K} are Hilbert spaces is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. We will write $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H}, \mathcal{H})$, and denote the range and kernel space of an operator T by $\text{ran } T$ and $\ker T$, respectively. If \mathcal{M} is a submanifold of \mathcal{H} then $\overline{\mathcal{M}}$ is the norm closure of \mathcal{M} and \mathcal{M}^\perp is the orthogonal complement of \mathcal{M} . Denote by $P_{\mathcal{M}}$ the orthogonal projection of \mathcal{H} onto \mathcal{M} when \mathcal{M} is closed. For an arbitrary subalgebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$ and any collection \mathcal{L} of closed subspaces of \mathcal{H} , $\text{Lat}(\mathcal{A})$ means the lattice of invariant subspaces of \mathcal{A} while $\text{Alg}(\mathcal{L})$ means the set of those $A \in \mathcal{B}(\mathcal{H})$ such that $A\mathcal{M} \subset \mathcal{M}$ for all $\mathcal{M} \in \mathcal{L}$. For any algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and $1 \leq n \leq \infty$, define the algebra $\mathcal{A}^{(n)}$ by

$$\mathcal{A}^{(n)} = \{T^{(n)} = \underbrace{T \oplus \cdots \oplus T}_{n \text{ times}} : T \in \mathcal{A}\}$$

which acts on the Hilbert space

$$\mathcal{H}^{(n)} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}}.$$

Given an operator T in $\mathcal{B}(\mathcal{H})$, the unital algebras generated by T in the weak operator topology and in the weak* topology are denoted by \mathcal{W}_T and \mathcal{A}_T , respectively. For any

subset $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, \mathcal{S}' is the set of operators commuting with every elements of \mathcal{S} and called the commutant of \mathcal{S} . A von Neumann algebra \mathcal{A} is a unital C^* -algebra contained in $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology, i.e., a unital C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A}'' = \mathcal{A}$ in light of the double commutant theorem. The von Neumann algebra generated by T , i.e., the smallest von Neumann algebra containing T will be denoted by \mathcal{W}_T^* . An important consequence of the double commutant theorem is the equality

$$\mathcal{W}_N^* = \{N\}'' ,$$

where N is a normal operator.

If μ is a compactly supported, regular Borel measure on \mathbb{C} and $N_\mu : L^2(\mu) \rightarrow L^2(\mu)$ is defined by $(N_\mu f)(z) = zf(z)$, $f \in L^2(\mu)$, then N_μ is a normal operator. Given a σ -finite measure space $(\mathcal{X}, \Omega, \mu)$ and function $\phi \in L^\infty(\mu)$, let M_ϕ be the multiplication operator $M_\phi f = \phi f$ on $L^2(\mu)$. If $\mathcal{A}_\mu = \{M_\phi : \phi \in L^\infty(\mu)\}$ then \mathcal{A}_μ is a von Neumann algebra with the property $\mathcal{A}_\mu = \mathcal{A}'_\mu = \mathcal{A}''_\mu$. If μ is assumed to be compactly supported on \mathbb{C} then $\{N_\mu\}' = \mathcal{A}_\mu$. If \mathcal{H} is separable and $N \in \mathcal{B}(\mathcal{H})$ is a normal operator then there exist mutually singular measures $\mu_\infty, \mu_1, \mu_2, \dots$ (some of which may be zero measures) such that N is unitarily equivalent to

$$N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus \dots .$$

If Δ_n denotes the support of the measure μ_n , $n = 1, 2, \dots, \infty$, the function $m_N : \mathbb{C} \rightarrow \{0, 1, \dots, \infty\}$ associated with the normal operator N defined by $m_N = \infty \cdot \chi_{\Delta_\infty} + \chi_{\Delta_1} + 2\chi_{\Delta_2} + \dots$ is a Borel function and called the multiplicity function of N . A normal operator is said to have uniform finite multiplicity if its multiplicity function is finite a.e. on \mathbb{C} .

Denote by H^p , $0 < p \leq \infty$, the usual Hardy spaces on the unit disc \mathbb{D} . For two functions θ and θ' in H^∞ , we say that θ divides θ' or θ' is a divisor of θ , denoted by $\theta|\theta'$, if $\theta' = \theta\phi$ for some $\phi \in H^\infty$. If θ and θ' differ only by a constant scalar factor of absolute value one, i.e., $\theta|\theta'$ and $\theta'|\theta$, then we use the notation $\theta \equiv \theta'$. For a family F of functions in H^∞ , the notation $\bigwedge F$ (or $\bigwedge_i f_i$ if $F = \{f_i : i \in I\}$, or $f_1 \wedge f_2$ if $F = \{f_1, f_2\}$) stands for the greatest common inner divisor of F . The least common inner multiple of F is denoted by $\bigvee F$ (or $\bigvee_i f_i$ if $F = \{f_i : i \in I\}$, or $f_1 \vee f_2$ if $F = \{f_1, f_2\}$).

For any inner function θ , the space $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$ is an invariant subspace for S^* , the adjoint of the unilateral shift S on the Hardy space H^2 . The operator $S(\theta) \in \mathcal{B}(\mathcal{H}(\theta))$ defined by the requirement that $S(\theta)^* = S^*|_{\mathcal{H}(\theta)}$ is called a Jordan block. A contraction $T \in \mathcal{B}(\mathcal{H})$ is called completely nonunitary if it does not have any nontrivial unitary direct summand. For a completely nonunitary contraction T , the Sz.-Nagy–Foias functional calculus is an algebra homomorphism $u \mapsto u(T) \in \mathcal{B}(\mathcal{H})$ of the algebra H^∞ which extends the usual polynomial calculus. For instant given any $u \in H^\infty$, $u(S)$ is the analytic Toeplitz operator on H^2 , i.e., the multiplication operator by u , and $u(S(\theta)) = P_{\mathcal{H}(\theta)} u(S)|_{\mathcal{H}(\theta)}$.

A completely nonunitary contraction $T \in \mathcal{B}(\mathcal{H})$ is said to be of class C_0 if $u(T) = 0$ for some nonzero function $u \in H^\infty$. If T is of class C_0 , the ideal $\{u \in H^\infty : u(T) = 0\}$ is of the form $m_T H^\infty$, where m_T is an inner function, uniquely determined up to a constant factor of absolute value one, and called the minimal function of T . The operator $S(\theta)$ is one of operators of class C_0 and its minimal function is θ . One of the important things about $S(\theta)$ is

$$\{S(\theta)\}' = \mathcal{W}_{S(\theta)} = \{u(S(\theta)) : u \in H^\infty\}.$$

Any invariant subspace $\mathcal{M} \subset \mathcal{H}(\theta)$ for $S(\theta)$ is of the form $\mathcal{M} = \phi H^2 \ominus \theta H^2$ where ϕ divides θ .

An operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is a quasiaffinity if it is an injection with dense range, i.e., if it has a (possibly unbounded) inverse defined on a dense domain in \mathcal{H}' . An operator $T \in \mathcal{B}(\mathcal{H})$ is called a quasiaffine transform of $T' \in \mathcal{B}(\mathcal{H}')$ if there exists a quasiaffinity $Q \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ satisfying $T'Q = QT$ which is denoted by $T \prec T'$. If $T \prec T'$ then $T'^* \prec T^*$. The operators T and T' are called quasisimilar, denoted by $T \sim T'$, if $T \prec T'$ and $T' \prec T$. If $T_1 \sim T_2$ and $T_2 \sim T_3$ then $T_1 \sim T_3$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$ the cyclic multiplicity μ_T is the smallest cardinal of a subset $\mathcal{M} \subset \mathcal{H}$ with the property that $\bigvee_{n=0}^\infty T^n \mathcal{M} = \mathcal{H}$, where the symbol \bigvee is closed linear span, while T is said to have finite cyclic multiplicity if $\mu_T < \infty$. Quasissimilarity plays an important role in classification of operators of class C_0 . Every operator T of class C_0 is quasissimilar to a unique Jordan operator, i.e., to an operator of the form

$$\bigoplus_i S(\theta_i)$$

where the values of i are ordinal numbers and the inner functions θ_i are subject to the conditions that $\theta_i \equiv 1$ for some $i \geq 0$, $\theta_{i_2} | \theta_{i_1}$ whenever $i_1 \leq i_2$, and $\theta_{i_1} \equiv \theta_{i_2}$ if $\text{card}(i_1) = \text{card}(i_2)$. The properties of operators of class C_0 stated in this section are known in the literature. For more details about such operators, the reader may consult [3] and [7].

For reader's convenience, in the following theorem we state the facts about operators of class C_0 that will be frequently used in the sequel. We refer to [3, Proposition 2.4.9] for (1), [3, Corollary 3.1.7] for (2), [3, Theorem 3.1.16] for (3), [3, Theorem 3.5.1] for (4) and (5), [3, Corollary 3.5.10] for (6), [3, Theorem 4.1.2], and [3, Corollary 4.1.6] for (7).

Theorem 2.1. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $T' \in \mathcal{B}(\mathcal{H}')$ are operators of class C_0 and that $\theta, \theta' \in H^\infty$ are two inner functions.*

- (1) *An operator of the form $v(T)$, $v \in H^\infty$, is injective if and only if $v \wedge m_T \equiv 1$, in which case $v(T)$ is a quasiaffinity.*
- (2) *For any inner function θ , the adjoint $S(\theta)^*$ is unitarily equivalent to $S(\theta^\sim)$, where $\theta^\sim \in H^\infty$ is the function defined by $\theta^\sim(z) = \overline{\theta(\bar{z})}$, $z \in \mathbb{D}$.*
- (3) *Suppose $A \in \mathcal{B}(\mathcal{H}(\theta), \mathcal{H}(\theta'))$. Then $S(\theta')A = AS(\theta)$ if and only if $A = P_{\mathcal{H}(\theta')}u(S)|\mathcal{H}(\theta)$ where $u \in H^\infty$ such that $\theta' | u\theta$.*
- (4) *We have $T \prec T'$ if and only if $T' \prec T$.*

- (5) If T has finite cyclic multiplicity, then $T \sim \bigoplus_{j=0}^{n-1} S(\theta_j)$ where inner functions $\theta_0, \dots, \theta_{n-1}$ satisfy the condition $\theta_{j+1} | \theta_j$ for all j and $m_T = \theta_0$.
- (6) Suppose that \mathcal{H} and \mathcal{H}' are separable and $\bigoplus_{j=0}^{\infty} S(\theta_j)$ is the Jordan model of T . If $m_{T'} | \theta_j$ for all j then $\bigoplus_{j=0}^{\infty} S(\theta_j)$ is also the Jordan model of $T \oplus T'$.
- (7) We have

$$\{T\}'' = \{T\}' \cap \text{AlgLat}(T) = \mathcal{A}_T = \mathcal{W}_T.$$

Moreover, there exists a function $v \in H^\infty$ with the properties that $v(T)$ is a quasiaffinity and $\mathcal{W}_T = \{v(T)^{-1}u(T) \in \mathcal{B}(\mathcal{H}) : u \in H^\infty\}$.

3. CLOSABILITY PROPERTY AND UNIFORM FINITE MULTIPLICITY

Definition 3.1. Suppose that \mathcal{A} is a subalgebra of $\mathcal{B}(\mathcal{H})$. A densely defined linear transformation with domain $\mathcal{D}(X)$ is said to commute with the algebra \mathcal{A} if $\mathcal{D}(X)$ is dense in \mathcal{H} and invariant under \mathcal{A} , and $XT = TX$ on $\mathcal{D}(X)$ for any $T \in \mathcal{A}$, i.e., if its graph

$$\mathcal{G}(X) = \{h \oplus Xh : h \in \mathcal{D}(X)\}$$

is invariant for $\mathcal{A}^{(2)}$.

Let \mathcal{H} and \mathcal{K} be Hilbert spaces and let X and Y be densely defined linear transformations from \mathcal{H} into \mathcal{K} . Then Y is called an extension of X if $\mathcal{G}(X) \subset \mathcal{G}(Y)$ and in symbols this is denoted by $X \subset Y$. The linear transformation X is called closed if its graph is closed in $\mathcal{H} \oplus \mathcal{K}$, while it is called closable if the closure $\overline{\mathcal{G}(X)}$ of its graph is a graph of some linear transformation which is denoted by \overline{X} and called the closure of X . In fact, X is closable if and only if for any sequence $\{h_n \oplus Xh_n\}$ in $\mathcal{G}(X)$ which converges to $0 \oplus k$ as $n \rightarrow \infty$ it follows that $k = 0$. The following lemma gives basic facts about certain products of bounded operators and densely defined linear transformations.

Lemma 3.2. Let A be any operator and X a densely defined linear transformation.

- (1) If A is injective and AX is closable then X is closable. Particularly, if $AX \subset B$ for some operator B then X is closable.
- (2) If XA is also a densely defined linear transformation and X is closable, then XA is closable.
- (3) Let \mathcal{H} and \mathcal{K} be Hilbert spaces and set $W = P_{\mathcal{H}}|_{\mathcal{H} \oplus \mathcal{K}}$. If the domain of X is dense in \mathcal{H} and XW is closable, then X is closable.

Proof. Pick any sequence $\{h_n \oplus Xh_n\}$ in $\mathcal{G}(X)$ which converges to $0 \oplus k$ as $n \rightarrow \infty$. If A is injective and AX is closable then $h_n \oplus AXh_n \in \mathcal{G}(AX)$ and $h_n \oplus AXh_n \rightarrow 0 \oplus (Ak)$ as $n \rightarrow \infty$ which shows that $Ak = 0$, and so $k = 0$. Note that AX is closable if $AX \subset B$ for some operator B . Hence (1) follows. Next, pick any sequence $\{h_n \oplus XAh_n\}$ in $\mathcal{G}(XA)$ which converges to $0 \oplus k$ as $n \rightarrow \infty$. Since the sequence $\{Ah_n \oplus XAh_n\}$ is in $\mathcal{G}(X)$ which converges to $0 \oplus k$ as $n \rightarrow \infty$, it follows that $k = 0$ if X is closable. Hence (2) holds. By examining the graph $\mathcal{G}(XW) = \mathcal{G}(X) \oplus \mathcal{K}$, it is easy to see that (3) holds. \square

Definition 3.3. An unital algebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$ is said to have the closability property if every densely defined linear transformation commuting with \mathcal{A} is closable.

Suppose that $(\mathcal{X}, \Omega, \mu)$ is a measure space. Recall that any function in $L^2(\mu)$ is the quotient of two bounded functions. If $f \in L^2(\mu)$, for instance, define

$$(3.1) \quad v_f = \begin{cases} 1/f & \text{if } |f| > 1 \\ 1 & \text{if } |f| \leq 1 \end{cases} \quad \text{and} \quad u_f = f v_f.$$

Then u_f and v_f are in $L^\infty(\mu)$, v_f is nonzero μ -a.e., and $f = u_f/v_f$.

To investigate the main topic of this section, we need some lemmas.

Lemma 3.4. Suppose that $(\mathcal{X}, \Omega, \mu)$ is a σ -finite measure space, and let $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$ where $\mathcal{H}_j = L^2(\mu)$, $j = 1, 2, \dots, n$, and $n < \infty$. If X is a densely defined linear transformation with dense domain $\mathcal{D}(X)$ commuting with the von Neumann algebra $\mathcal{A}_\mu^{(n)}$ then the manifold

$$\mathcal{D}(X) \cap [\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{j-1} \oplus \{0\} \oplus \mathcal{H}_{j+1} \oplus \dots \oplus \mathcal{H}_n]$$

is dense in $\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{j-1} \oplus \{0\} \oplus \mathcal{H}_{j+1} \oplus \dots \oplus \mathcal{H}_n$ for $j = 1, 2, \dots, n$.

Proof. It suffices to show that the manifold

$$\mathcal{D} = \mathcal{D}(X) \cap [\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-1} \oplus \{0\}]$$

is dense in $\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-1} \oplus \{0\}$. By taking linear combinations of elements in \mathcal{D} , it is equivalent to showing the claim that for any $j = 1, 2, \dots, n-1$, functions of the form $0 \oplus \dots \oplus 0 \oplus h_j \oplus 0 \oplus \dots \oplus 0$, $h_j \in \mathcal{H}_j$, can be approximated by elements of \mathcal{D} . Consequently, it is enough to show that the claim holds when $j = 1$. To show this, first assume $\mu(\mathcal{X}) < \infty$ and pick two sequences $f_m = \bigoplus_{j=1}^n f_{j,m}$ and $g_m = \bigoplus_{j=1}^n g_{j,m}$ in $\mathcal{D}(X)$ so that $f_m \rightarrow 1 \oplus 0 \oplus \dots \oplus 0$ and $g_m \rightarrow 0 \oplus \dots \oplus 0 \oplus 1$ in \mathcal{H} . For any m , let $v_{f_{n,m}}, u_{f_{n,m}}, v_{g_{n,m}}$, and $u_{g_{n,m}}$ be the functions defined as in (3.1). Then simple computations show that

$$\begin{aligned} k_m &:= \left(\bigoplus_{j=1}^n M_{v_{f_{n,m}} u_{g_{n,m}}} \right) f_m - \left(\bigoplus_{j=1}^n M_{u_{f_{n,m}} v_{g_{n,m}}} \right) g_m \\ &= \left(\bigoplus_{j=1}^{n-1} (v_{f_{n,m}} u_{g_{n,m}} f_{j,m} - u_{f_{n,m}} v_{g_{n,m}} g_{j,m}) \right) \oplus 0, \end{aligned}$$

and hence the vector k_m belongs to \mathcal{D} for all m since its last term in the summand is 0. Using the properties that $v_{f_{n,m}}, u_{g_{n,m}} \rightarrow 1$ and $u_{f_{n,m}} \rightarrow 0$ in $L^2(\mu)$ as $m \rightarrow \infty$, together with the fact that $\|v_f\|_\infty, \|u_f\|_\infty \leq 1$ for any $f \in L^2(\mu)$ yields that k_m converges to $1 \oplus 0 \oplus \dots \oplus 0$ in \mathcal{H} . Note that for any $\phi \in L^\infty(\mu)$, $(\bigoplus_{j=1}^n M_\phi) k_m$ is a sequence in \mathcal{D} which converges to $\phi \oplus 0 \oplus \dots \oplus 0$ in \mathcal{H} as $m \rightarrow \infty$. Since $L^\infty(\mu)$ is dense in $L^2(\mu)$, it is clear that the claim holds for $j = 1$.

We next consider the general case, i.e., \mathcal{X} is countable union of disjoint sets $\{\Delta_l\}_{l=1}^\infty$ for which $\mu(\Delta_l) < \infty$ for all l . Set $\mathcal{K}_l = \bigoplus_{j=1}^n L^2(\mu|_{\Delta_l})$, $l \geq 1$, where the measure $\mu|_{\Delta_l}$ gives the same values as μ does on Δ_l and 0 off Δ_l and any function in $L^2(\mu|_{\Delta_l})$

is viewed as a function defined on \mathcal{X} which vanishes off Δ_l . With this identification, it is easy to see that the identity

$$\mathcal{D}_l := \left(\bigoplus_{j=1}^n M_{\chi_{\Delta_l}} \right) \mathcal{D}(X) = \mathcal{D}(X) \cap \mathcal{K}_l$$

holds and that \mathcal{D}_l is a dense manifold of \mathcal{K}_l for all l . This yields that $X|_{\mathcal{K}_l}$ is also a densely defined linear transformation with dense domain \mathcal{D}_l commuting with $\mathcal{A}_{\mu|\Delta_l}^{(n)}$, and consequently

$$\mathcal{D}_l \cap \left[\left(\bigoplus_{j=1}^{n-1} L^2(\mu|\Delta_l) \right) \oplus \{0\} \right]$$

is dense in $(\bigoplus_{j=1}^{n-1} L^2(\mu|\Delta_l)) \oplus \{0\}$ by the first part of the proof. For any $\epsilon > 0$ and any $f \in (\bigoplus_{j=1}^{n-1} \mathcal{H}_j) \oplus \{0\}$, let N be a sufficiently large integer such that $\|f - P_{\bigoplus_{l=1}^N \mathcal{K}_l} f\|_{\mathcal{H}} < \epsilon$. For any $l = 1, 2, \dots, N$, pick an element $x_l \in \mathcal{D}_l \cap [(\bigoplus_{j=1}^{n-1} L^2(\mu|\Delta_l)) \oplus \{0\}]$ so that $\|x_l - P_{\mathcal{K}_l} f\|_{\mathcal{H}} < \epsilon/N$. Then it is clear that $x = \sum_{l=1}^N x_l \in \mathcal{D}$ and

$$\|x - f\|_{\mathcal{H}} \leq \sum_{l=1}^N \|x_l - P_{\mathcal{K}_l} f\|_{\mathcal{H}} + \|f - P_{\bigoplus_{l=1}^N \mathcal{K}_l} f\|_{\mathcal{H}} < 2\epsilon.$$

Since ϵ is arbitrary, the lemma follows. \square

Lemma 3.5. *Under the same assumptions of Lemma 3.4, the manifold*

$$\mathcal{D}(X) \cap [\{0\} \oplus \dots \oplus \{0\} \oplus \mathcal{H}_j \oplus \{0\} \oplus \dots \oplus \{0\}]$$

is dense in $\{0\} \oplus \dots \oplus \{0\} \oplus \mathcal{H}_j \oplus \{0\} \oplus \dots \oplus \{0\}$ for $j = 1, 2, \dots, n$.

Proof. It suffices to show that the assertion holds when $j = 1$. Set $\mathcal{K} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-1} \oplus \{0\}$, and consider the linear transformation $Y = P_{\mathcal{K}} X|_{\mathcal{K}} : \mathcal{D}(Y) \subset \mathcal{K} \rightarrow \mathcal{K}$ with domain denoted by $\mathcal{D}(Y)$. Then Y is densely defined since $\mathcal{D}(Y) = \mathcal{D}(X) \cap \mathcal{K}$ is dense in \mathcal{K} by Lemma 3.5. Moreover, Y commutes with the algebra $\mathcal{A}_{\mu}^{(n-1)} \oplus \{0\}$. Indeed, for any $\phi \in L^{\infty}(\mu)$ and $h \in \mathcal{D}(Y)$, we have $((\bigoplus_{j=1}^{n-1} M_{\phi}) \oplus 0)h = (\bigoplus_{j=1}^n M_{\phi})h \in \mathcal{D}(Y)$ and

$$\begin{aligned} Y \left(\left(\bigoplus_{j=1}^{n-1} M_{\phi} \right) \oplus 0 \right) h &= P_{\mathcal{K}} X \left(\bigoplus_{j=1}^n M_{\phi} \right) h \\ &= P_{\mathcal{K}} \left(\bigoplus_{j=1}^n M_{\phi} \right) Xh \\ &= \left(\left(\bigoplus_{j=1}^{n-1} M_{\phi} \right) \oplus 0 \right) P_{\mathcal{K}} Xh = \left(\left(\bigoplus_{j=1}^{n-1} M_{\phi} \right) \oplus 0 \right) Yh. \end{aligned}$$

Applying Lemma 3.4 to Y implies that

$$\begin{aligned} &\mathcal{D}(Y) \cap [\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-2} \oplus \{0\} \oplus \{0\}] \\ &= \mathcal{D}(X) \cap [\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-2} \oplus \{0\} \oplus \{0\}] \end{aligned}$$

is dense in $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{n-2} \oplus \{0\} \oplus \{0\}$. Continuing the same method up to finite times implies the desired result. The proof is complete. \square

Lemma 3.6. *Under the same assumptions of Lemma of 3.4, if $X = 0$ on the sets $\mathcal{D}(X) \cap [\{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{H}_j \oplus \{0\} \oplus \cdots \oplus \{0\}]$, $j = 1, 2, \dots, n$, then $X \equiv 0$ on $\mathcal{D}(X)$.*

Proof. We will finish the proof by showing the claim that $X = 0$ on the set

$$\mathcal{D}_m := \mathcal{D}(X) \cap [\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \oplus \{0\} \oplus \cdots \oplus \{0\}]$$

for any $m = 1, 2, \dots, n$. Obviously, the claim is true if $m = 1$. Suppose that the claim holds for $m \leq j$, where $j \leq n - 1$. Applying Lemma 3.5 to find two nonzero elements $\Phi = \phi_1 \oplus \cdots \oplus \phi_{j+1} \oplus 0 \oplus \cdots \oplus 0$ and $\Psi = 0 \oplus \cdots \oplus 0 \oplus \psi \oplus 0 \oplus \cdots \oplus 0$ in $\mathcal{D}(X)$ where $\psi \in \mathcal{H}_{j+1}$. We may assume that $\phi_1, \dots, \phi_{j+1}, \psi$ are all bounded by multiplying them by a bounded function. Then $X\Psi = 0$ and

$$\begin{aligned} \left(\bigoplus_{k=1}^n M_{\psi} \right) X\Phi &= \left(\bigoplus_{k=1}^n M_{\psi} \right) X\Phi - \left(\bigoplus_{k=1}^n M_{\phi_{j+1}} \right) X\Psi \\ &= X \left[\left(\bigoplus_{k=1}^n M_{\psi} \right) \Phi - \left(\bigoplus_{k=1}^n M_{\phi_{j+1}} \right) \Psi \right] \\ &= X[(\phi_1\psi) \oplus \cdots \oplus (\phi_j\psi) \oplus 0 \oplus \cdots \oplus 0] \\ &= 0 \end{aligned}$$

because $X = 0$ on \mathcal{D}_j . For any μ -measurable set Δ with finite measure, making use of Lemma 3.5 and the preceding identity yields that

$$(3.2) \quad \left(\bigoplus_{k=1}^n M_{\chi_{\Delta}} \right) X\Phi = 0,$$

whence

$$(3.3) \quad X\Phi = 0.$$

Indeed, first pick a sequence of functions $\{f_m\}$ in \mathcal{H}_{j+1} so that $0 \oplus \cdots \oplus 0 \oplus f_m \oplus 0 \oplus \cdots \oplus 0$ belongs to $\mathcal{D}(X)$ for all m and $f_m \rightarrow \chi_{\Delta}$ in $L^2(\mu)$ as $m \rightarrow \infty$ by Lemma 3.5. Note that

$$\begin{aligned} &\left(\bigoplus_{k=1}^n M_{\chi_{\Delta} v_{f_m}} \right) (0 \oplus \cdots \oplus 0 \oplus f_m \oplus 0 \oplus \cdots \oplus 0) \\ &= 0 \oplus \cdots \oplus 0 \oplus \chi_{\Delta} u_{f_m} \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{D}(X) \end{aligned}$$

and $\chi_{\Delta} u_{f_m}$ converges to χ_{Δ} as $m \rightarrow \infty$, where functions v_{f_m} and u_{f_m} are defined as in (3.1). Then replacing ψ by $\chi_{\Delta} u_{f_m}$ and letting $m \rightarrow \infty$ yield (3.2). Finally, for any $h = h_1 \oplus \cdots \oplus h_{j+1} \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{D}(X)$, pick a nonzero function $v \in L^{\infty}(\mu)$ with the property that $vh_1, \dots, vh_{j+1} \in L^{\infty}(\mu)$. Then it follows from (3.3) that

$$\left(\bigoplus_{m=1}^n M_v \right) Xh = X \left[\left(\bigoplus_{m=1}^n M_v \right) h \right] = 0,$$

and consequently $Xh = 0$ by the injectivity of $\bigoplus_{m=1}^n M_v$. Hence $X = 0$ on \mathcal{D}_{j+1} and the claim holds by induction. This completes the proof. \square

It follows from [4, Lemma 3.2] that if X is a densely defined linear transformation commuting with \mathcal{A}_μ where μ is a probability measure then there exists an everywhere defined measurable function k such that $Xf = kf$ for every f in the domain of X . As a result, there exists a function $v \in L^\infty(\mu)$ which is nonzero almost everywhere such that $a = vk \in L^\infty(\mu)$ and $M_v X \subset M_a$. Combining the proof of Lemma 3.4 with the above conclusions, it is easy to extend Arveson's results to σ -finite case.

Theorem 3.7. *Suppose that $(\mathcal{X}, \Omega, \mu)$ is a σ -finite measure space. Then the von Neumann algebra $\mathcal{A}_\mu^{(n)}$ has the closability property if and only if n is finite.*

Proof. The closability property of the algebra $\mathcal{A}_\mu^{(n)}$ yields that it has uniform finite multiplicity, i.e., n is finite by [2, Proposition 3.5(5)]. Conversely, suppose $n < \infty$ and let X be a densely defined linear transformation with domain $\mathcal{D}(X)$ commuting with $\mathcal{A}_\mu^{(n)}$. Set $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$, where $\mathcal{H}_j = L^2(\mu)$ for all j , and

$$\mathcal{D}_j = \mathcal{D}(X) \cap [\{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{H}_j \oplus \{0\} \oplus \cdots \oplus \{0\}].$$

Then Lemma 3.5 implies that there exist densely defined linear transformations $X_{ij} : \mathcal{D}_j \subset \mathcal{H}_j \rightarrow \mathcal{H}_i$, $i, j = 1, 2, \dots, n$, such that

$$X(0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0) = (X_{1j} f_j) \oplus \cdots \oplus (X_{nj} f_j)$$

for any $0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{D}_j$. It is apparent that X_{ij} commutes with the algebra \mathcal{A}_μ for $i, j = 1, \dots, n$. Indeed, if $\phi \in L^\infty(\mu)$ then

$$\begin{aligned} (M_\phi X_{1j} f_j) \oplus \cdots \oplus (M_\phi X_{nj} f_j) &= \left(\bigoplus_{j=1}^n M_\phi \right) X(0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0) \\ &= X \left(\bigoplus_{j=1}^n M_\phi \right) (0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0) \\ &= X(0 \oplus \cdots \oplus 0 \oplus (\phi f_j) \oplus 0 \oplus \cdots \oplus 0) \\ &= (X_{1j} M_\phi f_j) \oplus \cdots \oplus (X_{nj} M_\phi f_j), \end{aligned}$$

which shows that $M_\phi X_{ij} \subset X_{ij} M_\phi$. Let $v \in L^\infty(\mu)$ be a nonzero function so that $M_v X_{ij} \subset M_{a_{ij}}$, where $a_{ij} \in L^\infty(\mu)$ for $i, j = 1, 2, \dots, n$, and set

$$A = (M_{a_{ij}}), \quad B = \bigoplus_{j=1}^n M_v, \quad \text{and} \quad Z = BX - A.$$

Since A and B are in the algebra $M_{n \times n}(\mathcal{A}_\mu) = \{\mathcal{A}_\mu^{(n)}\}'$, it follows that Z is a densely defined linear transformation with domain $\mathcal{D}(X)$ commuting with $\mathcal{A}_\mu^{(n)}$. Moreover, the preceding discussions show that $Z = 0$ on the sets $\mathcal{D}_1, \dots, \mathcal{D}_n$ from which we deduce that $Z = 0$ on $\mathcal{D}(X)$ or, equivalently, $BX \subset A$, by Lemma 3.6. By virtue of Lemma 3.2(1), X is closable since B is injective, and consequently $\mathcal{A}_\mu^{(n)}$ has the closability property. The proof is complete. \square

Recall that an algebra $\mathcal{A}_1 \subset \mathcal{B}(\mathcal{H}_1)$ is a quasiaffine transform of an algebra $\mathcal{A}_2 \subset \mathcal{B}(\mathcal{H}_2)$ if there exists a quasiaffinity $Q \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ so that for any $T_2 \in \mathcal{A}_2$, we have $T_2 Q = Q T_1$ for some $T_1 \in \mathcal{A}_1$. It follows from [2, Proposition 5.2] that if \mathcal{A}_1 and \mathcal{A}_2 are unital algebras, \mathcal{A}_1 is a quasiaffine transformation of \mathcal{A}_2 , and if \mathcal{A}_2 has the closability property, then so does \mathcal{A}_1 . Particularly, this says that the closability property is invariant under unitary equivalence.

Theorem 3.8. *Suppose that \mathcal{H} is a separable Hilbert space and N is a normal operator on \mathcal{H} . Then the von Neumann algebra \mathcal{W}_N^* generated by N has the closability property if and only if N has uniform finite multiplicity.*

Proof. First note that there exist mutually singular measures $\mu_\infty, \mu_1, \mu_2, \mu_3, \dots$ (some of which may be zero) such that N is unitarily equivalent to

$$N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus N_{\mu_3}^{(3)} \oplus \dots,$$

and therefore

$$\mathcal{W}_N^* \cong \mathcal{A}_{\mu_\infty}^{(\infty)} \oplus \mathcal{A}_{\mu_1} \oplus \mathcal{A}_{\mu_2}^{(2)} \oplus \mathcal{A}_{\mu_3}^{(3)} \oplus \dots.$$

Since the closability property is invariant under unitary equivalence, by virtue of [2, Lemma 3.8] and Theorem 3.7 it follows that \mathcal{W}_N^* has the closability property if and only if N has uniform finite multiplicity. \square

4. UNBOUNDED LINEAR MAPS INTERTWINING OPERATORS OF CLASS C_0

Recall that an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to intertwine operators $T_2 \in \mathcal{B}(\mathcal{H}_2)$ and $T_1 \in \mathcal{B}(\mathcal{H}_1)$ if $AT_1 = T_2A$. Denote by $\mathcal{I}(T_1, T_2)$ the set of all operators in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ intertwining T_2 and T_1 . Interest in this section is primarily looking at densely defined linear transformations that intertwine operators of class C_0 as the definition is given below.

Definition 4.1. *Suppose that $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$. A densely defined linear transformation X with domain $\mathcal{D}(X)$ is said to intertwine operators T_2 and T_1 if $\mathcal{D}(X)$ is dense in \mathcal{H}_1 and invariant under T_1 , and $T_2X = XT_1$ on $\mathcal{D}(X)$.*

It is apparent that a densely defined linear transformation X intertwines operators T_2 and T_1 if and only if its graph $\mathcal{G}(X)$ is an invariant manifold for $T_1 \oplus T_2$. Let $J : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_2 \oplus \mathcal{H}_1$ be the isomorphism defined by $J(h_1 \oplus h_2) = (-h_2) \oplus h_1$. If X intertwines operators T_2 and T_1 then $J\mathcal{G}(X)$ is invariant for $T_2 \oplus T_1$ from which we infer that $\mathcal{G}(X^*) = [J\mathcal{G}(X)]^\perp$ is invariant for $T_2^* \oplus T_1^*$ or, equivalently, X^* intertwines T_1^* and T_2^* . In addition, if X is closable then its closure \overline{X} also intertwines T_2 and T_1 . If X intertwines T and T for any T in an algebra \mathcal{A} , then X commutes with \mathcal{A} .

We next consider some special types of densely defined linear transformations. Let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be any two operators. If $A \in \mathcal{I}(T_1, T_2)$ and $B \in \{T_1\}'$ is a quasiaffinity then AB^{-1} is a densely defined linear transformation intertwining T_2

and T_1 since the graph $\mathcal{G}(AB^{-1}) = \{Bh_1 \oplus Ah_1 : h_1 \in \mathcal{H}_1\}$ is invariant for $T_1 \oplus T_2$. Further, it is apparent that $\mathcal{G}(AB^{-1}) = \text{ran } M$, where

$$M = \begin{pmatrix} B & 0 \\ A & 0 \end{pmatrix} \in \{T_1 \oplus T_2\}'.$$

Hence we have

$$\begin{aligned} \mathcal{G}((AB^{-1})^*) &= [J\mathcal{G}(AB^{-1})]^\perp \\ &= [J\text{ran } M]^\perp = J[\text{ran } M]^\perp = J\ker M^* \\ &= \{h_2 \oplus h_1 \in \mathcal{H}_2 \oplus \mathcal{H}_1 : A^*h_2 = B^*h_1\} \\ &= \mathcal{G}(B^{*-1}A^*), \end{aligned}$$

which shows that $(AB^{-1})^* = B^{*-1}A^*$. Therefore, AB^{-1} is closable if and only if $B^{*-1}A^*$ is densely defined in which case $\overline{AB^{-1}} = (B^{*-1}A^*)^*$. In general, we have

$$\begin{aligned} J^*\mathcal{G}(B^{*-1}A^*) &= \{h_1 \oplus h_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2 : B^*h_1 + A^*h_2 = 0\} \\ &= \ker M^*, \end{aligned}$$

or, equivalently, $\mathcal{G}(B^{*-1}A^*) = J\ker M^*$ whence $[J^*\mathcal{G}(B^{*-1}A^*)]^\perp = \overline{\text{ran } M} = \overline{\mathcal{G}(AB^{-1})}$.

If T is an operator of class C_0 and X is a closed, densely defined linear transformation commuting with T then $X = \overline{AB^{-1}}$ for some operator $A \in \{T\}'$ and quasiaffinity $B \in \{T\}'$ which was prove by Bervocivi [1]. The following result, though not stated explicitly, is essentially due to him.

Proposition 4.2. *Let T_1 and T_2 be two operators of class C_0 . If X is a closed, densely defined linear transformation intertwining T_2 and T_1 then $X = \overline{AB^{-1}} = (B^{*-1}A^*)^*$, where $A \in \mathcal{I}(T_1, T_2)$ and $B \in \{T_1\}'$ is a quasiaffinity. Consequently, $\mathcal{G}(X) = \overline{\text{ran } M}$, where*

$$M = \begin{pmatrix} B & 0 \\ A & 0 \end{pmatrix} \in \{T_1 \oplus T_2\}'.$$

Proof. Denote by $\mathcal{D}(X)$ the domain of X . Let $T = T_1 \oplus T_2|_{\mathcal{G}(X)}$ and define the quasiaffinity $Q : \mathcal{G}(X) \rightarrow \mathcal{H}_1$ by $Q(h_1 \oplus Xh_1) = h_1$, $h_1 \in \mathcal{D}(X)$. It is clear that $T_1Q = QT$ by the hypothesis, and hence $T \prec T_1$. This implies that $TQ' = Q'T_1$ for some quasiaffinity $Q' \in \mathcal{B}(\mathcal{H}_1, \mathcal{G}(X))$ by Theorem 2.1(4). Assume $Q'h_1 = Bh_1 \oplus Ah_1$, $h_1 \in \mathcal{H}_1$. Then it is easy to see that $B \in \{T_1\}'$ and $T_2A = AT_1$. If $Bh_1 = 0$ for some $h_1 \in \mathcal{H}_1$, then $Q'h_1 = 0$ since $\mathcal{G}(X)$ is a graph, and hence $h_1 = 0$. The fact that $Q'\mathcal{H}_1$ is dense in $\mathcal{G}(X)$ yields that $\mathcal{D}(X) \subset \overline{B\mathcal{H}_1}$, and hence B has dense range. Finally, the rest desired results follow from the equalities

$$\overline{\mathcal{G}(AB^{-1})} = \overline{\{Bh \oplus Ah : h \in \mathcal{H}_1\}} = \overline{Q'\mathcal{H}_1} = \mathcal{G}(X)$$

and the fact that AB^{-1} is closable. The proof is complete. \square

By considering the adjoint X^* of X which intertwines operators of class C_0 , the preceding proposition implies the next result.

Proposition 4.3. *Let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be any two operators of class C_0 . If X is a closed, densely defined linear transformation intertwining T_2 and T_1*

then $X = B^{-1}A = (A^*B^{*-1})^*$, where $A \in \mathcal{I}(T_1, T_2)$ and $B \in \{T_2\}'$ is a quasiaffinity. Consequently, $\mathcal{G}(X) = J^*\ker M$, where

$$M = \begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix} \in \{T_2 \oplus T_1\}'$$

and J is the isomorphism defined by $J(h_1 \oplus h_2) = (-h_2) \oplus h_1$, $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$.

Proof. Since $\mathcal{G}(X^*)$ is an invariant subspace for the operator $T_2^* \oplus T_1^*$ of class C_0 , applying the conclusions in Proposition 4.2 to the closed, densely defined linear transformation X^* yields that $X^* = \overline{A^*B^{*-1}}$ for some $A^* \in \mathcal{I}(T_2^*, T_1^*)$ and quasiaffinity $B^* \in \{T_2^*\}'$. Since A^*B^{*-1} is closable, it follows that $X = X^{**} = \overline{A^*B^{*-1}}^* = (A^*B^{*-1})^* = B^{-1}A$. The identity $\mathcal{G}(X) = J^*\ker M$ is easy to verify. The proposition follows. \square

Corollary 4.4. *If T is an operator of class C_0 and X is a closed densely defined linear transformation commuting with T or the algebra \mathcal{W}_T then $X = B^{-1}A$ where $A, B \in \{T\}'$ and B is a quasiaffinity.*

Proof. The first assertion follows immediately from Proposition 4.3. To finish the proof, it suffices to show that $B^{-1}A$ also commutes with \mathcal{W}_T . Based the facts that $\mathcal{W}_T = \{T\}''$ which is a part of Theorem 2.1(7) and $\mathcal{D}(B^{-1}A) = \{h \in \mathcal{H} : Ah \in \text{ran } B\}$, it is easy to see that $\mathcal{G}(B^{-1}A)$ is an invariant subspace for $\mathcal{W}_T^{(2)}$. The corollary follows. \square

Let T_1 and T_2 be any operators and suppose that $A \in \mathcal{I}(T_1, T_2)$ and $B \in \{T_2\}'$ is injective. In general, $B^{-1}A$ is a closed linear transformation that intertwines T_2 and T_1 but not necessarily densely defined. If T_1 and T_2 are of class C_0 then $B^{-1}A$ is densely defined if and only if $BC = AD$ for some operator $C \in \mathcal{I}(T_1, T_2)$ and quasiaffinity $D \in \{T_1\}'$ by Proposition 4.2. Note that in general the condition $BC = AD$ only implies that $\overline{CD^{-1}} \subset B^{-1}A$. To precede further, we need the following lemma to verify when two closed, densely defined linear transformation are equal.

Lemma 4.5. *Suppose that T_1 and T_2 are any operators and that A, C are in $\mathcal{I}(T_1, T_2)$ and $B, D \in \{T_2\}'$ are quasiaffinities. If $EB = FD$ for some quasiaffinities $E, F \in \{T_2\}'$ then $B^{-1}A = D^{-1}C$ if and only if $EA = FC$. Particularly, if $BD = DB$ then $B^{-1}A = D^{-1}C$ if and only if $BC = DA$.*

Proof. Note that $B^{-1}A = (EB)^{-1}(EA)$ and $D^{-1}C = (FD)^{-1}(FC) = (EB)^{-1}(FC)$. Then it is apparent that $B^{-1}A = D^{-1}C$ if and only if $EA = FC$. If $BD = DB$ then letting $E = D$ and $F = B$ yields the desired result. \square

For operators of class C_0 with finite multiplicity, we have the following proposition.

Proposition 4.6. *Suppose that T_1 and T_2 are any two operators of class C_0 with $\mu_{T_2} < \infty$. If X is a closed, densely defined linear transformation intertwining T_2 and T_1 then $X = v(T_2)^{-1}A$, where $A \in \mathcal{I}(T_1, T_2)$ and $v \in H^\infty$ so that $v(T_2)$ is a quasiaffinity.*

Proof. By Proposition 4.3, $X = B^{-1}A_0$ for some $A_0 \in \mathcal{I}(T_1, T_2)$ and quasiaffinity $B \in \{T_2\}'$. Since T_2 has finite multiplicity, it follows from [1, Proposition 2] that $BC = v(T_2)$ where $C \in \{T_2\}'$ and $v \in H^\infty$ so that $v(T_2)$ and C are quasiaffinities. Let $A = CA_0$. Then the fact that B commutes with $v(T_2)$, together with the identity $BA = v(T_2)A_0$, yields that $B^{-1}A_0 = v(T_2)^{-1}A$ by Lemma 4.5. This finishes the proof. \square

Proposition 4.7. *Suppose that T_1 and T_2 are operators of class C_0 and that $A \in \mathcal{I}(T_1, T_2)$ and $v \in H^\infty$ so that $v \wedge (m_{T_1} \vee m_{T_2}) \equiv 1$. Then $X = Av(T_1)^{-1}$ is a closable, densely defined linear transformation intertwining $u(T_2)$ and $u(T_1)$ for any $u \in H^\infty$ and its closure is $\overline{X} = v(T_2)^{-1}A$.*

Proof. It is clear that X intertwines $u(T_2)$ and $u(T_1)$ for any $u \in H^\infty$, and that $v(T_2)X \subset A$ whence X is closable by Theorem 2.1(1) and Lemma 3.2. Since \overline{X} intertwines T_2 and T_1 , from Proposition 4.3 we infer that $\overline{X} = D^{-1}C$ for some $C \in \mathcal{I}(T_1, T_2)$ and quasiaffinity $D \in \{T_2\}'$ which shows that $DA = Cv(T_1) = v(T_2)C$. Hence $\overline{X} = v(T_2)^{-1}A$ by Lemma 4.5. This completes the proof. \square

If the additional assumption $\mu_{T_1} < \infty$ is put in Proposition 4.6, more results can be obtained.

Proposition 4.8. *Suppose that T_1 and T_2 are operators of class C_0 with finite multiplicity. If X is a closed, densely defined linear transformation intertwining T_2 and T_1 then $X = \overline{Av(T_1)^{-1}} = v(T_2)^{-1}A$ for some $A \in \mathcal{I}(T_1, T_2)$ and $v \in H^\infty$ so that $v(T_1)$ and $v(T_2)$ are quasiaffinities.*

Proof. The second equality in the assertion holds by Proposition 4.7. Next, suppose $T_1 \in \mathcal{B}(\mathcal{H}_1)$, $T_2 \in \mathcal{B}(\mathcal{H}_2)$ and denote by $\mathcal{D}(X)$ by the domain of X . Then the closed, densely defined linear transformation

$$Y = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} : \mathcal{D}(X) \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$$

commutes with $T_1 \oplus T_2$ since the graph $\mathcal{G}(Y) = \{(h_1 \oplus h_2) \oplus (0 \oplus Xh_1) : h_1 \in \mathcal{D}(X), h_2 \in \mathcal{H}_2\}$ is an invariant subspace for $(T_1 \oplus T_2) \oplus (T_1 \oplus T_2)$. By virtue of Proposition 4.6 and Theorem 2.1(1), we conclude that $Y = v(T_1 \oplus T_2)^{-1}M$ where $v \in H^\infty$ so that $v \wedge (m_{T_1} \vee m_{T_2}) \equiv 1$ and $M \in \{T_1 \oplus T_2\}'$. This shows that $X = v(T_2)^{-1}A$ for some operator $A \in \mathcal{I}(T_1, T_2)$, as desired. \square

For nonconstant inner functions θ, ϕ and functions $u, v \in H^\infty$ so that $\phi|u\theta$ and $v \wedge (\theta \vee \phi) \equiv 1$, the operator $A = P_{\mathcal{H}(\phi)}u(S)|\mathcal{H}(\theta)$ is in $\mathcal{I}(S(\theta), S(\phi))$ and $v(S(\theta)), v(S(\phi))$ are quasiaffinities by Theorem 2.1(1) and (2). Thus, by Proposition 4.7 and 4.8 we have proved the following corollary.

Corollary 4.9. *Let θ and ϕ be nonconstant inner functions.*

(1) *For any $u, v \in H^\infty$ with $\phi|u\theta$ and $v \wedge (\theta \vee \phi) \equiv 1$, the linear transformation $X = P_{\mathcal{H}(\phi)}u(S)v(S(\theta))^{-1}$ is a closable, densely defined linear transformation intertwining $u(S(\phi))$ and $u(S(\theta))$ for any $u \in H^\infty$ and its closure is*

$$\overline{X} = v(S(\phi))^{-1}P_{\mathcal{H}(\phi)}u(S)|\mathcal{H}(\theta).$$

(2) Any closed, densely defined linear transformation X intertwining $S(\phi)$ and $S(\theta)$ is of the form

$$X = \overline{P_{\mathcal{H}(\phi)}u(S)v(S(\theta))^{-1}} = v(S(\phi))^{-1}P_{\mathcal{H}(\phi)}u(S)|_{\mathcal{H}(\theta)},$$

where v and u are in H^∞ so that $v \wedge (\theta \vee \phi) \equiv 1$ and $\phi|u\theta$.

We finish this section with the following theorem which will be used in Section 5.

Theorem 4.10. *Let θ and ϕ be nonconstant inner functions. If X is a densely defined linear transformation intertwining $u(S(\phi))$ and $u(S(\theta))$ for any $u \in H^\infty$ then X is closable with closure*

$$\overline{X} = v(S(\phi))^{-1}u(S(\phi))P_{\mathcal{H}(\phi)}|_{\mathcal{H}(\theta)},$$

where $u, v \in H^\infty$ so that $\phi|u\theta$ and $v \wedge (\theta \vee \phi) \equiv 1$.

Proof. Let $\theta' \equiv \theta \vee \phi$, $\phi' = \theta'/\phi$, and $\mathcal{H}(\theta)^\perp = \mathcal{H}(\theta') \ominus \mathcal{H}(\theta)$. Consider the linear transformation $Y = \phi'(S(\theta'))XW$ where $W = P_{\mathcal{H}(\theta)}|_{\mathcal{H}(\theta')}$. First note that the domain of Y is $\mathcal{D}(X) \oplus \mathcal{H}(\theta)^\perp$ which is dense in $\mathcal{H}(\theta')$ where $\mathcal{D}(X)$ as before denotes the domain of X . Then for any $h \in \mathcal{D}(X)$, $g \in \mathcal{H}(\theta)^\perp$, and $u' \in H^\infty$ we have

$$\begin{aligned} u'(S(\theta'))Y(h \oplus g) &= u'(S(\theta'))\phi'(S(\theta'))Xh \\ &= \phi'(S(\theta'))u'(S(\phi))Xh \\ &= \phi'(S(\theta'))Xu'(S(\theta))h \\ &= \phi'(S(\theta'))XP_{\mathcal{H}(\theta)}u'(S(\theta'))(h + g) \\ &= \phi'(S(\theta'))Yu'(S(\theta'))(h \oplus g) \end{aligned}$$

where we use the fact that $u'(S(\theta'))\phi'(S(\theta'))|_{\mathcal{H}(\phi)} = \phi'(S(\theta'))u'(S(\phi))$. Thus Y commutes with the algebra $\mathcal{W}_{S(\theta')}$ and further, by virtue of [2, Proposition 3.7] we derive that Y is closable. Since the operator $\phi'(S(\theta'))|_{\mathcal{H}(\phi)}$ is injective, it is apparent that X is closable by Lemma 3.2(1) and (3). The desired form of the closure \overline{X} follows from Corollary 4.9(2). \square

5. CLOSABILITY PROPERTY OF ALGEBRAS \mathcal{W}_T AND $H^\infty(T)$

In this section, we will give necessary and sufficient conditions for an operator T of class C_0 so that the algebras \mathcal{W}_T and $H^\infty(T) = \{u(T) : u \in H^\infty\}$ have the closability property. We first prove some lemmas.

Lemma 5.1. *Suppose that $\theta_0, \dots, \theta_n$ are nonconstant inner functions and $T = \bigoplus_{j=0}^n S(\theta_j)$. If X is a densely defined linear transformation with dense domain $\mathcal{D}(X)$ commuting with the algebra $H^\infty(T)$ then the manifold*

$$\mathcal{D}(X) \cap [\{0\} \oplus \dots \oplus \{0\} \oplus \mathcal{H}(\theta_j) \oplus \{0\} \oplus \dots \oplus \{0\}]$$

is dense in $\{0\} \oplus \dots \oplus \{0\} \oplus \mathcal{H}(\theta_j) \oplus \{0\} \oplus \dots \oplus \{0\}$ for any $j = 0, 1, \dots, n$.

Proof. It is enough to show that the manifold

$$\mathcal{D}_0 = \mathcal{D}(X) \cap [\mathcal{H}(\theta_0) \oplus \{0\} \oplus \dots \oplus \{0\}]$$

is dense in $\mathcal{H}(\theta_0) \oplus \{0\} \oplus \cdots \oplus \{0\}$. Let $\mathcal{H} = \bigoplus_{j=0}^n \mathcal{H}(\theta_j)$. First observe that the set of functions in $\bigoplus_{j=0}^n H^2$ whose projections onto \mathcal{H} belong to $\mathcal{D}(X)$ is a dense manifold of $\bigoplus_{j=0}^n H^2$. Using this observation to find sequences $\{f_m^{(0)}\}_{m=1}^\infty, \{f_m^{(1)}\}_{m=1}^\infty, \dots, \{f_m^{(n)}\}_{m=1}^\infty$ in $\bigoplus_{j=0}^n H^2$ so that $P_{\mathcal{H}} f_m^{(i)} \in \mathcal{D}(X)$ for all i, m and

$$\begin{aligned} f_m^{(i)} &:= f_m^{(i,0)} \oplus f_m^{(i,1)} \oplus \cdots \oplus f_m^{(i,n)} \\ &\rightarrow e_i \end{aligned}$$

as $m \rightarrow \infty$ for any $i = 0, 1, \dots, n$, where e_i is the standard basis in $\bigoplus_{j=0}^n H^2$ with the constant function 1 in the i -th entry and 0 elsewhere. Let $\Phi_m = (f_m^{(i,j)})$ be the matrix composed of the entries $f_m^{(i,j)}$ and let $a_m^{(i)}$ be the product of $(-1)^i$ and the determinant of the matrix obtained by deleting the first column and i -th row of Φ_m . Then it follows from linear algebra that

$$(5.1) \quad \sum_{i=0}^n a_m^{(i)} f_m^{(i)} = (\det \Phi_m) \oplus 0 \oplus \cdots \oplus 0 \rightarrow 1 \oplus 0 \oplus \cdots \oplus 0$$

in $H^{1/(n+1)}$ as $m \rightarrow \infty$ which implies the greatest common inner divisor of the inner parts of the functions $\{\det \Phi_m\}$ is 1. Indeed, if $h = \bigwedge_m (\det \Phi_m)$, then (5.1) implies that the limit function $1 \in hH^{1/(n+1)}$, and hence $h \equiv 1$. Next pick a bounded outer function v so that the function $v f_m^{(i,j)}$ is in H^∞ for all i, j , and m . Note that $v^n a_m^{(i)} \in H^\infty$, and therefore we have that $(v^n a_m^{(i)})(T) P_{\mathcal{H}} f_m^{(i)} \in \mathcal{D}(X)$ for all i and m by the hypothesis. Moreover, from (5.1) we deduce that

$$\begin{aligned} h_m &:= \sum_{i=0}^n (v^n a_m^{(i)})(T) P_{\mathcal{H}} f_m^{(i)} \\ &= \sum_{i=0}^n P_{\mathcal{H}} v^n(T) (a_m^{(i)} f_m^{(i)}) \\ &= (P_{\mathcal{H}(\theta_0)}(v^n \det \Phi_m)) \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{D}_0 \end{aligned}$$

for all m . Since

$$(5.2) \quad \bigwedge_{m=1}^\infty v^n \det \Phi_m \equiv 1$$

by the preceding discussion, it follows that

$$P_{\mathcal{H}(\theta_0)}(v^n \det \Phi_m) \neq 0$$

for some m , and consequently $h_m \neq 0$ for some m . Hence \mathcal{D}_0 contains a nonzero element from which we derive that the closure $\overline{\mathcal{D}_0} = \mathcal{H}(\theta_0) \oplus \{0\} \oplus \cdots \oplus \{0\}$. Indeed, since the closure $\overline{\mathcal{D}_0}$ is an invariant subspace for $S(\theta_0) \oplus 0 \oplus \cdots \oplus 0$ there exists some inner divisor ϕ_0 of θ_0 such that

$$\overline{\mathcal{D}_0} = (\phi_0 H^2 \ominus \theta_0 H^2) \oplus \{0\} \oplus \cdots \oplus \{0\}$$

which shows that $\phi_0 \equiv 1$ by (5.2). This completes the proof. \square

Lemma 5.2. *Under the same assumptions of Lemma 5.1, if $X = 0$ on the sets $\mathcal{D}(X) \cap [\{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{H}(\theta_j) \oplus \{0\} \oplus \cdots \oplus \{0\}]$, $j = 0, 1, \dots, n$, then $X \equiv 0$ on $\mathcal{D}(X)$.*

Proof. We will finish the proof by showing the claim that $X = 0$ on the sets

$$\mathcal{M}_m := \mathcal{D}(X) \cap [\mathcal{H}(\theta_0) \oplus \mathcal{H}(\theta_1) \oplus \cdots \oplus \mathcal{H}(\theta_m) \oplus \{0\} \oplus \cdots \oplus \{0\}], \quad m = 0, 1, \dots, n.$$

The case $m = 0$ follows from the hypothesis. Suppose now that the claim holds for $m \leq j$, where $j \leq n - 1$. Using lemma 5.1 to find two nonzero elements $f = P_{\mathcal{H}}(u_0 \oplus \cdots \oplus u_{j+1} \oplus 0 \oplus \cdots \oplus 0)$ and $g = P_{\mathcal{H}}(0 \oplus \cdots \oplus 0 \oplus u \oplus 0 \oplus \cdots \oplus 0)$ in $\mathcal{D}(X)$, where $u, u_0, \dots, u_{j+1} \in H^2$. By multiplying functions u, u_0, \dots, u_{j+1} by a bounded function in H^2 , we may assume $u, u_0, \dots, u_{j+1} \in H^\infty$. Since $Xg = 0$, it follows that

$$\begin{aligned} u(T)Xf &= u(T)Xf - u_{j+1}(T)Xg \\ &= X(u(T)f - u_{j+1}(T)g) \\ &= XP_{\mathcal{H}}[(uu_0) \oplus \cdots \oplus (uu_j) \oplus 0 \oplus \cdots \oplus 0] \\ &= 0, \end{aligned}$$

which yields that

$$(5.3) \quad XP_{\mathcal{H}}(u_0 \oplus \cdots \oplus u_{j+1} \oplus 0 \oplus \cdots \oplus 0) = 0$$

for any vector $P_{\mathcal{H}}(u_0 \oplus \cdots \oplus u_{j+1} \oplus 0 \oplus \cdots \oplus 0) \in \mathcal{D}(X)$ with $u_0, \dots, u_{j+1} \in H^\infty$ by Lemma 5.1. Finally, for any vector $f = P_{\mathcal{H}}(u_0 \oplus \cdots \oplus u_{j+1} \oplus 0 \oplus \cdots \oplus 0) \in \mathcal{D}(X)$ let v be a bounded outer function so that vu_0, \dots, vu_{j+1} are bounded. Then (5.3) shows that

$$\begin{aligned} v(T)Xf &= Xv(T)f \\ &= XP_{\mathcal{H}}[(vu_0) \oplus \cdots \oplus (vu_{j+1}) \oplus 0 \oplus \cdots \oplus 0] \\ &= 0. \end{aligned}$$

Since v is an outer function, $v(T)$ is a quasiaffinity, and hence $Xf = 0$. Then the claim is proved by induction and this completes the proof. \square

We now can prove the first main result of this section.

Theorem 5.3. *Let $\theta_0, \dots, \theta_n$ be nonconstant inner functions and let $T = \bigoplus_{j=0}^n S(\theta_j)$. Then the algebra $H^\infty(T)$ has the closability property.*

Proof. Let $\mathcal{H} = \bigoplus_{j=0}^n \mathcal{H}(\theta_j)$ and X be a densely defined linear transformation with domain $\mathcal{D}(X)$ commuting with the algebra $H^\infty(T)$. By virtue of Lemma 5.1, the set

$$\mathcal{D}_j := \mathcal{D}(X) \cap [\{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{H}(\theta_j) \oplus \{0\} \oplus \cdots \oplus \{0\}]$$

is dense in $\{0\} \oplus \cdots \oplus \{0\} \oplus \mathcal{H}(\theta_j) \oplus \{0\} \oplus \cdots \oplus \{0\}$ for any $j = 0, 1, \dots, n$. If each \mathcal{D}_j is viewed as a densely manifold of $\mathcal{H}(\theta_j)$, there exist densely defined linear transformations $X_{ij} : \mathcal{D}_j \subset \mathcal{H}(\theta_j) \rightarrow \mathcal{H}(\theta_i)$, $0 \leq i, j \leq n$, such that

$$X(0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0) = \bigoplus_{i=0}^n (X_{ij}f_j)$$

for all $0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{D}_j$. It is apparent that X_{ij} intertwines $u(S(\theta_i))$ and $u(S(\theta_j))$ for all $u \in H^\infty$ and $i, j = 0, \dots, n$. Indeed, we have $u(S(\theta_j))\mathcal{D}_j = u(T)\mathcal{D}_j \subset \mathcal{D}_j$ and

$$\begin{aligned} \bigoplus_{i=0}^n (X_{ij}u(S(\theta_j))f_j) &= X(0 \oplus \cdots \oplus 0 \oplus (u(S(\theta_j))f_j) \oplus 0 \oplus \cdots \oplus 0) \\ &= Xu(T)(0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0) \\ &= u(T)X(0 \oplus \cdots \oplus 0 \oplus f_j \oplus 0 \oplus \cdots \oplus 0) \\ &= \bigoplus_{i=0}^n (u(S(\theta_i))X_{ij}f_j), \end{aligned}$$

which implies that

$$X_{ij}u(S(\theta_j))f_j = u(S(\theta_i))X_{ij}f_j$$

for any $i, j = 0, 1, \dots, n$. In light of Theorem 4.10, there exist functions $v_0, v_1, \dots, v_n \in H^\infty$ and $A_{ij} \in \mathcal{I}(S(\theta_j), S(\theta_i))$, $i, j = 0, \dots, n$, with the properties that $v_i \wedge \theta_i \equiv 1$ and $v_i(S(\theta_i))X_{ij} \subset A_{ij}$ for any $i, j = 0, 1, \dots, n$. Let $A = (A_{ij})$, $B = \bigoplus_{j=0}^n v_j(S(\theta_j))$, and $Z = BX - A$. Notice that $A, B \in \{T\}'$ and B is injective. Then the above arguments show that Z is a densely defined linear transformation with domain $\mathcal{D}(X)$ commuting with $H^\infty(T)$ and vanishes on \mathcal{D}_j for any j . It follows from Lemma 5.2 that $Z = 0$ on $\mathcal{D}(X)$, and hence $BX \subset A$. Thus X is closable by Lemma 3.2(1), and consequently $H^\infty(T)$ has the closability property. This completes the proof. \square

By Theorem 5.3, together with the fact that the closability property is preserved under quasisimilarity, we have the following result.

Corollary 5.4. *For any operator T of class C_0 with finite multiplicity, the algebra $H^\infty(T)$ has the closability property.*

Proof. Since T is quasisimilar to its Jordan model $\bigoplus_{j=0}^n S(\theta_j)$ where n is finite and the algebra $H^\infty(\bigoplus_{j=0}^n S(\theta_j))$ has the closability property by Theorem 5.3, the algebra $H^\infty(T)$ has the closability property as well by [2, Proposition 5.2(2)]. \square

The following proposition is a direct consequence of Theorem 5.3 and [2, Proposition 3.5(5)].

Proposition 5.5. *For any nonconstant inner function θ , the algebra $H^\infty(S(\theta)^{(n)})$ has the closability property if and only if $n < \infty$.*

We will make use of the following lemma to investigate the closability property of $H^\infty(T)$ when T has infinity multiplicity.

Lemma 5.6. *Suppose that T_1 and T_2 are completely nonunitary contraction. If the algebra $H^\infty(T_1 \oplus T_2)$ has the closability property then so do the algebras $H^\infty(T_1)$ and $H^\infty(T_2)$.*

Proof. Suppose $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. For any densely defined linear transformation X with domain $\mathcal{D}(X)$ commuting with the algebra $H^\infty(T_1)$,

$$Y = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{D}(X) \oplus \mathcal{H}_2 \subset \mathcal{H} \rightarrow \mathcal{H}$$

is a densely defined linear transformation commuting with the algebra $H^\infty(T_1 \oplus T_2)$. If $H^\infty(T_1 \oplus T_2)$ has the closability property then Y is closable whence X is closable. Indeed, if X is regarded as a linear map from \mathcal{H}_1 into \mathcal{H} then $Y = XP_{\mathcal{H}_1}|_{\mathcal{H}}$, and so X is closable by Lemma 3.2(3). Hence $H^\infty(T_1)$, as well as $H^\infty(T_2)$, has the closability property. \square

Theorem 5.7. *Suppose that T is an operator of class C_0 acting on a separable Hilbert space. If $\bigoplus_{j=0}^\infty S(\theta_j)$ is the Jordan model of T , where θ_j is nonconstant for all j , then the algebra $H^\infty(T)$ has the closability property if and only if $\bigwedge_{j=0}^\infty \theta_j \equiv 1$.*

Proof. Let $\theta = \bigwedge_{j=0}^\infty \theta_j$. First assume that the algebra $H^\infty(T)$ has the closability property. Since $\theta|\theta_j$ for any j , it follows from Theorem 2.1(6) that $\bigoplus_{j=0}^\infty S(\theta_j)$ is also the Jordan model of $T \oplus T'$ where $T' = S(\theta)^{(\infty)}$ which shows that $T \sim T \oplus T'$. By [2, Proposition 5.2(2)], the algebra $H^\infty(T \oplus T')$ also has the closability property. But Lemma 5.6 yields that the algebra $H^\infty(T')$ has the closability property, which is a contradiction unless $\theta \equiv 1$ by Proposition 5.5.

Conversely, by [2, Proposition 5.2(2)] we may assume that $T = \bigoplus_{j=0}^\infty S(\theta)$ with $\theta \equiv 1$ and let X be any densely defined linear transformation with domain $\mathcal{D}(X)$ commuting with the algebra $H^\infty(T)$. For any $m \geq 0$, let $T_m = \bigoplus_{j=0}^m S(\theta_j)$ and $\mathcal{H}_m = \bigoplus_{j=0}^m \mathcal{H}(\theta_j)$, and consider the linear transformation $Y_m = \theta_m(T)X\theta_m(T) : \mathcal{D}(Y_m) \subset \mathcal{H}_m \rightarrow \mathcal{H}_m$ with domain

$$\mathcal{D}(Y_m) = \{h_m \in \mathcal{H}_m : \theta_m(T)h_m \in \mathcal{D}(X)\}.$$

Observe that if $h \in \mathcal{D}(X)$ then $\theta_m(T)P_{\mathcal{H}_m}h = \theta_m(T)h \in \mathcal{D}(X)$, which shows that $P_{\mathcal{H}_m}\mathcal{D}(X) \subset \mathcal{D}(Y_m)$ is dense in \mathcal{H}_m , and hence Y_m is densely defined. Moreover, Y_m commutes with the algebra $H^\infty(T_m)$, and consequently is closable by Theorem 5.3. Indeed, for any $h_m \in \mathcal{D}(Y_m)$ and $u \in H^\infty$ we have

$$\theta_m(T)u(T_m)h_m = \theta_m(T)u(T)h_m = u(T)\theta_m(T)h_m \in \mathcal{D}(X)$$

which shows that $u(T_m)h_m \in \mathcal{D}(Y_m)$, and

$$\begin{aligned} Y_mu(T_m)h_m &= \theta_m(T)X\theta_m(T)u(T_m)h_m \\ &= \theta_m(T)Xu(T)\theta_m(T)h_m \\ &= u(T)\theta_m(T)X\theta_m(T)h_m \\ &= u(T_m)\theta_m(T)X\theta_m(T)h_m = u(T_m)Y_mh_m. \end{aligned}$$

Finally, suppose $\{f_n \oplus Xf_n\}$ is a sequence in $\mathcal{G}(X)$ so that $f_n \oplus Xf_n \rightarrow 0 \oplus g$ as $n \rightarrow \infty$. Then for any fixed m we have $P_{\mathcal{H}_m}f_n \rightarrow 0$ and

$$Y_mP_{\mathcal{H}_m}f_n = \theta_m(T)X\theta_m(T)f_n = \theta_m(T)^2Xf_n \rightarrow \theta_m(T)^2g$$

as $n \rightarrow \infty$, which implies that $\theta_m(T)^2 g = 0$ for any $m \geq 0$ or, equivalently,

$$(5.4) \quad \theta_j | \theta_m^2 g_j \quad \text{for any } m, j \geq 0$$

if $g = \bigoplus_{j=0}^{\infty} g_j$, $g_j \in \mathcal{H}(\theta_j)$. It follows that $\theta_j | g_j$ for all j , i.e., $g = 0$. Indeed, for any fixed $n, j \geq 0$ and for any $m \geq \max\{n, j\}$, by (5.4) we have $(\theta_j/\theta_m) | \theta_n g_j$, and so $\bigvee_{m \geq n, j} (\theta_j/\theta_m) | \theta_n g_j$ which is equivalent to $\theta_j | \theta_n g_j$ since $\bigvee_{m \geq n, j} (\theta_j/\theta_m) \equiv \theta_j / (\bigwedge_{m \geq n, j} \theta_m) \equiv \theta_j$. Hence we proved that $\theta_j | \theta_n g_j$ for any $n, j \geq 0$. Repeating the above argument yields the desired result that $\theta_j | g_j$ for any j . Therefore, X is closable, and so the algebra $H^\infty(T)$ has the closability property. This completes the proof. \square

Assume now that $T \in \mathcal{B}(\mathcal{H})$ is an operator of class C_0 with the Jordan model $\bigoplus_i S(\theta_i)$. If

$$(5.5) \quad \bigwedge_{j < \omega} \theta_j \equiv 1$$

then $\theta_\omega \equiv 1$, and so $\bigoplus_i S(\theta_i) = \bigoplus_{j < \omega} S(\theta_j)$ and the underlying Hilbert space \mathcal{H} is separable. Therefore, it follows from Theorem 5.7 that $H^\infty(T)$ has the closability property if (5.5) holds. Conversely, if $\theta = \bigwedge_{j < \omega} \theta_j$ then Theorem 2.1(7) shows that

$$\left(\bigoplus_{j < \omega} S(\theta_j) \right) \oplus S(\theta)^{(\infty)} \sim \bigoplus_{j < \omega} S(\theta_j),$$

and hence by comparison of Jordan models we have $T \oplus S(\theta)^{(\infty)} \sim T$. Hence it is easy to see from [2, Proposition 5.2(2)], Proposition 5.5, and Lemma 5.6 that (5.5) holds if the algebra $H^\infty(T)$ has the closability property. We summarize these conclusions in the following theorem.

Theorem 5.8. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an operator of class C_0 with the Jordan model $\bigoplus_i S(\theta_i)$. Then the algebra $H^\infty(T)$ has the closability property if and only if $\bigwedge_{j < \omega} S(\theta_j) \equiv 1$ in which case \mathcal{H} is separable.*

Corollary 5.9. *For any operator T of class C_0 , the algebra $H^\infty(T)$ has the closability property if and only if the algebra $H^\infty(T^*)$ has the closability property.*

Proof. If $H^\infty(T)$ has the closability property, the Jordan model of T must be of the form $\bigoplus_{j=0}^{\infty} S(\theta_j)$ with $\bigwedge_{j=0}^{\infty} \theta_j \equiv 1$ by Theorem 5.8. Note that we have $T^* \sim \bigoplus_{j=0}^{\infty} S(\theta_j^\sim)$ by Theorem 2.1(2) and $\bigwedge_j \theta_j^\sim \equiv 1$. As a consequence of Theorem 5.8 and [2, Proposition 5.2(2)], the algebra $H^\infty(T^*)$ has the closability property. The same is also true for the converse and the corollary is proved. \square

Recall that an operator T of class C_0 is said to have the property (P) if every injection $A \in \{T\}'$ is a quasiaffinity. It was shown in [3, Theorem 7.1.9] that T has the property (P) if and only if its Jordan model satisfies the condition (5.5). We next investigate the closability property of other algebras generated by an operator of class C_0 . Recall from Theorem 2.1(7) that

$$\{T\}'' = \{T\}' \cap \text{AlgLat}(T) = \mathcal{A}_T = \mathcal{W}_T = \mathcal{F}_T$$

for any operator T of class C_0 . By virtue of [2, Lemma 3.3], if $H^\infty(T)$ has the closability property then the algebra \mathcal{W}_T as well as $\text{AlgLat}(T)$ has the closability property (it is shown in [2] that the commutant $\{T\}'$ of any operator T of class C_0 has closability property). The following theorem states that the converse is also true.

Theorem 5.10. *For any operator T of class C_0 with Jordan model $\bigoplus_i S(\theta_i)$, the followings are equivalent:*

- (1) *the algebra $H^\infty(T)$ has the closability property;*
- (2) *$\bigwedge_{j < \omega} \theta_j \equiv 1$;*
- (3) *T has the property (P);*
- (4) *the algebra \mathcal{W}_T has the closability property.*

Proof. The equivalences of (1), (2), and (3) were proved. By [2, Lemma 3.3], it suffices to show that (4) implies (1), i.e., any densely defined linear transformation X commuting with the algebra $H^\infty(T)$ is closable if the algebra \mathcal{W}_T has the closability property. Suppose $T \in \mathcal{B}(\mathcal{H})$ and define $\mathcal{D} = \{Ah : A \in \mathcal{W}_T, h \in \mathcal{D}(X)\}$ where $\mathcal{D}(X)$ is the domain of X . By Theorem 2.1(7), let v be in H^∞ with the properties that $v(T)$ is a quasiaffinity and $\mathcal{W}_T = \{v(T)^{-1}u(T) \in \mathcal{B}(\mathcal{H}) : u \in H^\infty\}$. Then $\mathcal{D}(X) \subset \mathcal{D}$ is dense in \mathcal{H} and the domain of the linear transformation $Xv(T)$ contains \mathcal{D} . Indeed, for any $Ah \in \mathcal{D}$ we have $v(T)(Ah) = (Av(T))h \in \mathcal{D}(X)$ since $Av(T) \in H^\infty(T)$ which shows that $Ah \in \{k \in \mathcal{H} : v(T)k \in \mathcal{D}(X)\}$, the domain of $Xv(T)$. Moreover, the densely defined linear transformation $Y = Xv(T)|_{\mathcal{D}}$, the restriction of $Xv(T)$ to \mathcal{D} , commutes with \mathcal{W}_T whence Y is closable. Indeed, for any $A, B \in \mathcal{W}_T$ and $h \in \mathcal{D}(X)$ we have $ABh \in \mathcal{D}$ and

$$\begin{aligned} v(T)(YA)Bh &= v(T)X(v(T)AB)h = X(v(T)^2AB)h \\ &= X(v(T)A)(v(T)B)h = (v(T)A)X(v(T)B)h \\ &= v(T)(AY)Bh, \end{aligned}$$

where we use the facts that $v(T)AB \in H^\infty(T)$ and X commutes with $H^\infty(T)$. By the injectivity of $v(T)$, we infer from the above equalities that $(YA)Bh = (AY)Bh$. Finally, it will be shown that if $\{h_n\}$ is an arbitrary sequence in $\mathcal{D}(X)$ so that $h_n \oplus (Xh_n) \rightarrow 0 \oplus k$ as $n \rightarrow \infty$ then $k = 0$. Indeed, the fact that Y is closable shows that $Yh_n = Xv(T)h_n = v(T)Xh_n \rightarrow v(T)k = 0$ as $n \rightarrow \infty$, and consequently $k = 0$ by the injectivity of $v(T)$ again. This finishes the proof that X is closable. \square

For any operator T of class C_0 and $\mathcal{M} \in \text{Lat}(T)$, it was shown in [3, Corollary 7.1.17] that T has the property (P) if and only if $T|_{\mathcal{M}}$ and $P_{\mathcal{M}^\perp}T|_{\mathcal{M}^\perp}$ have the property (P). Hence we have the following theorem which generalizes Lemma 5.6.

Theorem 5.11. *Suppose that T is an operator of class C_0 and $\mathcal{M} \in \text{Lat}(T)$. If $T_1 = T|_{\mathcal{M}}$ and $T_2 = P_{\mathcal{M}^\perp}T|_{\mathcal{M}^\perp}$ then the algebra $H^\infty(T)$ has the closability if and only if the algebras $H^\infty(T_1)$ and $H^\infty(T_2)$ have the closability property.*

The proof of Lemma 5.6 combined with the conclusions in the preceding theorem and Proposition 4.3 yield the following corollary whose proof we omit.

Corollary 5.12. *Suppose that T_1 and T_2 are operators of class C_0 so that the algebras $H^\infty(T_1)$ and $H^\infty(T_2)$ have the closability property. Then any densely defined linear transformation X intertwining $u(T_2)$ and $u(T_1)$ for any $u \in H^\infty$ is closable. Moreover, $\overline{X} = B^{-1}A$ for some operator $A \in \mathcal{I}(T_1, T_2)$ and quasiaffinity $B \in \{T_1\}'$.*

The preceding corollary shows that for any $A \in \mathcal{I}(T_1, T_2)$ and injection $B \in \{T_1\}'$, AB^{-1} is a closable, densely defined linear transformation intertwining operators $u(T_2)$ and $u(T_1)$ for any $u \in H^\infty$ provided that $H^\infty(T_1 \oplus T_2)$ has the closability property. Note that $H^\infty(T_2^* \oplus T_1^*)$ also has the closability property. Consequently, $C^{-1}A$ is closed and densely defined for any quasiaffinity $C \in \{T_2\}'$ since $C^{-1}A = (A^*C^{*-1})^*$ and A^*C^{*-1} is closable.

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